Journal of Sound and Vibration (1999) **225**(4), 701–722
Article No. jsvi.1999.2268, available online at http://www.idealibrary.com on IDEAL®

FREE VIBRATIONS OF DOUBLY CURVED IN-PLANE NON-HOMOGENEOUS SHELLS

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(*Received* 19 *November* 1998, *and in final form* 23 *February* 1999)

A method for investigating free vibrations rectangular non-homogeneous shells is proposed. By non-homogeneity, we define a chage of stiffness of shell bending caused by an introduction of another material or change of a shell thickness is defined. It is assumed that a shell possesses an arbitrary value of rectangular parts with different bending stiffness.

1. INTRODUCTION

The important contributions to the theory of plates and shell dynamics have been brought in by Timoshenko and Voinovskij-Kreiger [1], Vlasov [2], Vol'mir [3], Bolotin [4], Filippov [5], Satchenkov [6], Krysko [7] and others. Many of the dynamical problems of vibration of conical shells with classical boundary conditions in a frame of both linear and non-linear theories have been solved. A review of plate and shell vibration theories for various boundary conditions and different shape of plates and shells has been given in monographs [8, 9], and for homogeneous plates and shells in the works of Liessa [10] and Cowper *et al.* [11].

It should be emphasized that vast literature on plate and shell vibration has recently been published. Among others, literature reviews on recent plate and shell vibrations can be found in references [12–14].

In reference [15–17], free vibrations in relation to the boundary conditions and geometrical parameters have been investigated using the Ritz method with higher approximations to the plates and shells. The open cylindrical shells dynamics has been analyzed in reference [18], and the trapezoidal plates have been analyzed in reference [19].

More literature on free vibration analysis using the Ritz method with higher approximations for plates are given in references [20–23].

In this paper the influence of bending stiffness, thickness change and the geometrical parameters, $k_x = k_y$ on the free vibrations of rectangular plates and shells have been investigated using the Bubnov–Galerkin method with higher approximations. As an example, a free support with unstretched (uncompressed) ribs has been analyzed. The doubly curved shells considered are constituted by isotropic material which shows in-plane non-homogeneity simply in the sense that Young's modulus is taken as a function of the in-plane shell co-ordinate $E = E(x, y)$.

2. ANALYSIS

2.1. PROBLEM DEFINITION

Consider a rectangular shell with $x \in [0, a]$, $y \in [0, b]$ and thickness with $2h\,(-h \leqslant z \leqslant h)$. An average surface is related to the rectangular co-ordinates with the origin located in the lower-left corner of the shell. The unit vectors of the co-ordinates form the right oriented three vectors.

The kinematic Kirchhoff–Love model is used. The deformations in an arbitrary point on the shell are defined by the relations

$$\varepsilon_{xx} = \varepsilon_{11} - z\,\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \varepsilon_{22} - z\,\frac{\partial^2 w}{\partial y^2}, \quad \varepsilon_{xy} = \varepsilon_{12} - 2z\,\frac{\partial^2 w}{\partial x\,\partial y}, \tag{1}$$

where

$$\varepsilon_{11} = \frac{\partial u}{\partial x} - k_x w, \quad \varepsilon_{22} = \frac{\partial v}{\partial y} - k_y w, \quad \varepsilon_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2}$$

with u, v denoting the average surface displacements, $w(x, y)$ a deflection, and k_x, k_y the curvatures.

The material of the shell is assumed to be isotropic, elastic and non-homogeneous governed by Hook's law

$$\varepsilon_{xx} = \frac{1}{E}\,(\sigma_{xx} - \mu\sigma_{yy}), \quad \varepsilon_{yy} = \frac{1}{E}\,(\sigma_{yy} - \mu\sigma_{xx}), \quad \varepsilon_{xy} = \frac{2(1+\mu)}{E}\,\sigma_{xy}, \tag{3}$$

where $E = E(x, y)$.

Integrating equation (3) along the thickness, we get the relationships between forces and deformations of the form

$$\varepsilon_{11} = \frac{1}{2hE}\,(T_{11} + \mu T_{22}), \quad \varepsilon_{22} = \frac{1}{2hE}\,(T_{22} + \mu T_{11}), \quad \varepsilon_{12} = \frac{1+\mu}{hE}\,T_{12}, \tag{4}$$

where

$$T_{11} = \int_{-h}^{h} \sigma_{xx}\,\mathrm{d}z, \quad T_{12} = \int_{-h}^{h} \sigma_{yy}\,\mathrm{d}z, \quad T_{12} = \int_{-h}^{h} \sigma_{xy}\,\mathrm{d}z.$$

Let us consider a movement process between the time moments t_0 and t_1. The true trajectories differ from the other possible trajectories because

$$\int_{t_2}^{t_1} (\delta K - \delta V - \delta' W)\, \mathrm{d}t = 0, \tag{5}$$

where K is the kinetic energy, V is the potential energy, and $\delta' W$ is the sum of elementary works of the external forces.

The potential energy

$$V = \frac{1}{2} \int_0^a \int_0^b \int_{-h}^{h} (\sigma_{xx}\, \varepsilon_{xx} + \sigma_{yy}\, \varepsilon_{yy} + \sigma_{xy}\varepsilon_{xy})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{6}$$

and the kinetic energy

$$K = \frac{1}{2} \int_0^a \int_0^b \frac{\rho}{g} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right\} \mathrm{d}x\, \mathrm{d}y \tag{7}$$

are given by equations (6) and (7), where $\rho = \rho(x, y)$ is the material density, and work of the external forces is defined by

$$\delta' W = \int_0^a \int_0^b [p_x\, \delta u + p_y \delta v + q \delta w]\, \mathrm{d}x\, \mathrm{d}y. \tag{8}$$

Substituting equations (1)–(3) into equations (6) and (7) we can formulate the variational equation. The latter gives the governing equations as well as the boundary and initial conditions. We introduce the force function

$$T_{11} = \frac{\partial^2 F}{\partial y^2}, \quad T_{22} = \frac{\partial^2 F}{\partial x^2}, \quad T_{12} = \frac{\partial^2 F}{\partial x \partial y}. \tag{9}$$

Then the variational equation has the form

$$\int_{t_0}^{t_1} \int_0^a \int_0^b \left\{ \left[\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + p_x - \frac{2h\rho}{g} \frac{\partial^2 u}{\partial t^2} \right] \delta u \right.$$

$$+ \left[\frac{\partial T_{22}}{\partial y} + \frac{\partial T_{12}}{\partial x} + p_y - \frac{2h\rho}{g} \frac{\partial^2 v}{\partial t^2} \right] \delta v$$

$$+ \left[\frac{2h^3 E}{3(1-\mu^2)} \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 (\cdot)}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 (\cdot)}{\partial y^2} + 2(1-\mu) \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 (\cdot)}{\partial x \partial y} \right. \right.$$

$$\left. \left. + \mu \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 (\cdot)}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 (\cdot)}{\partial x^2} \right) \right) - k_x \frac{\partial^2 F}{\partial y^2} - k_y \frac{\partial^2 F}{\partial x^2} + q - \frac{2h\rho}{g} \frac{\partial^2 w}{\partial t^2} \right] \delta w$$

$$+\left[\frac{a}{2h}\left(\left(\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$

$$\left.+\,2(1+\mu)\,\frac{\partial^2 F}{\partial x\partial y}\,\frac{\partial^2(\cdot)}{\partial x\partial y}\right)+k_x\,\frac{\partial^2 w}{\partial y^2}+k_y\,\frac{\partial^2 w}{\partial x^2}\Bigg]\,\delta F\Bigg\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+\int_0^a\int_0^b\frac{2h\rho}{g}\left[\frac{\partial u}{\partial t}\,\delta u+\frac{\partial v}{\partial t}\,\delta v+\frac{\partial w}{\partial t}\,\delta w\right]\Bigg|_{t_0}^{t_1}\mathrm{d}x\,\mathrm{d}y$$

$$-\int_t^{t_1}\int_0^a\left[T_{22}\,\delta v+T_{12}\,\delta u+\left(\varepsilon_{11}\,\frac{\partial(\cdot)}{\partial y}-\frac{\partial\varepsilon_{11}}{\partial y}+\frac{\partial\varepsilon_{12}}{\partial x}\right)\delta F\right]\Bigg|_0^b\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^{t_1}\int_0^b\left[T_{11}\,\delta u+T_{12}\,\delta v+\left(\varepsilon_{22}\,\frac{\partial(\cdot)}{\partial x}-\frac{\partial\varepsilon_{22}}{\partial x}-\varepsilon_{12}\,\frac{\partial(\cdot)}{\partial y}\right)\delta F\right]\Bigg|_0^a\mathrm{d}y\,\mathrm{d}t=0. \tag{10}$$

From equation (10) we get the equations of motion

$$\frac{\partial T_{11}}{\partial x}+\frac{\partial T_{12}}{\partial y}+p_x-\frac{2h\rho}{g}\,\frac{\partial^2 u}{\partial t^2}=0,$$

$$\frac{\partial T_{22}}{\partial y}+\frac{\partial T_{12}}{\partial x}+p_y-\frac{2h\rho}{g}\,\frac{\partial^2 v}{\partial t^2}=0,$$

$$\frac{2h^3}{3(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2(\cdot)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\,\frac{\partial^2 w}{\partial x\partial y}\,\frac{\partial^2(\cdot)}{\partial x\,\partial y}\right.$$

$$\left.+\,\mu\left(\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\,\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]-k_x\,\frac{\partial^2 F}{\partial y^2}-k_y\,\frac{\partial^2 F}{\partial x^2}+q-\frac{2h\rho}{g}\,\frac{\partial^2 w}{\partial t^2}=0, \tag{11}$$

and the deformation continuity relation

$$\frac{a_1}{2h}\left[\left(\frac{\partial^2 F}{\partial y^2}-\mu\,\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{\partial^2 F}{\partial x^2}-\mu\,\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$

$$\left.+\,2(1+\mu)\,\frac{\partial^2 F}{\partial x\partial y}\right)\frac{\partial^2(\cdot)}{\partial x\partial y}\Bigg]+k_x\,\frac{\partial^2 w}{\partial y^2}+k_y\,\frac{\partial^2 w}{\partial x^2}=0, \tag{12}$$

where $a_1(x,y)=1/E(x,y)$, and $(\cdot)$ denotes a variation of the being sought functions either $w(x,y)$ or $F(x,y)$.

In addition, we consider the transverse vibration without the occurrence of elastic waves along the co-ordinates x and y. Therefore, equations (11) are substantially simplified. Neglecting p_x and p_y, and introducing the force function according to formulae (9) from equations (11) the following hybrid type system is obtained: (a) one equilibrium equation

$$\frac{2h^3 E}{3(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right.$$
$$\left.+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]-k_x\frac{\partial^2 F}{\partial^2 y^2}-k_y\frac{\partial^2 F}{\partial x^2}+q-\frac{2h\rho}{g}\frac{\partial^2 w}{\partial t^2}=0; \quad (13)$$

(b) the deformation continuity equation

$$\frac{a_1}{2h}\left[\left(\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$
$$\left.+2(1+\mu)\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}=0. \quad (14)$$

When in equations (13) and (14) we take $E=\text{const}$, $\rho=\text{const}$ and using integration by parts we finally obtain the equations

$$\frac{D}{2h}\nabla^4 w=\nabla_k^2 F+\frac{q}{2h}-\frac{\rho}{g}\frac{\partial^2 w}{\partial t^2}=0,$$

$$\frac{1}{2hE}\nabla^4 F+\nabla_k^2 w=0, \quad (15)$$

where $\nabla_k^2(\cdot)=k_y(\partial^2/\partial x^2)+k_x(\partial^2/\partial y^2)$, $D=E(2h)^3/12(1-\mu^2)$ denotes the cylindrical stiffness, $E=E(x,y)$ the changeable Young's modulus and μ the Poisson coefficient.

For the purpose of numerical integration, equations (13) and (14) are reduced to the non-dimensional form using the relations (the non-dimensional parameters are denoted by bars)

$$w=2h\bar{w},\ x=a\bar{x},\ y=b\bar{y},\ F=E_0(2h)^3\bar{F},\ k_x=\frac{2h}{a^2}\bar{k}_x,\ k_y=\frac{2h}{b^2}\bar{k}_y,$$

$$\lambda=\frac{a}{b},\ q=\frac{(2h)^4E_0}{a^2b^2}\bar{q},\ t=\frac{ab}{2h}\sqrt{\frac{\rho_0}{gE_0}}\,\bar{t},\ E=E_0\bar{E},\ \rho=\rho_0\bar{\rho}. \quad (16)$$

The non-dimensional equations have the form (bars are omitted)

$$\frac{E}{12(1-\mu^2)}\left[\left(\lambda^{-2}\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial(\cdot)}{\partial x^2}+\left(\lambda^2\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}\right.$$
$$\left.+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]-k_x\frac{\partial^2 F}{\partial y^2}-k_y\frac{\partial^2 F}{\partial x^2}+q=\rho\frac{\partial^2 w}{\partial t^2},$$

$$a_1\left[\left(\lambda^2\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\lambda^{-2}\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$
$$\left.+\,2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}=0. \tag{17}$$

The shell bending stiffness is defined with the help of the unit Heaviside's function

$$E=1-\sum_{k=1}^{N}(1-\gamma_{1k})\gamma_{0k},\ \rho=1-\sum_{k=1}^{N}(1-\gamma_{2k})\gamma_{0k},\ a_1=1+\sum_{k=1}^{N}\left(\frac{1}{\gamma_{lk}}-1\right)\gamma_{0k},$$
$$\gamma_{0k}=\Gamma_0(x-x_{1k};y-y_{1k})-\Gamma_0(x-x_{2k};y-y_{1k})-\Gamma_0(x-x_{1k};y-y_{2k})$$
$$-\,\Gamma_0(x-x_{2k};y-y_{2k}). \tag{18}$$

Substituting equation (18) into equation (17) and taking $q=0$, the following equations in a hybrid form governing a free vibrations of non-homogeneous shells are obtained

$$\Phi_1\left(\frac{\partial^2 w}{\partial x^2},\frac{\partial^2 w}{\partial y^2},\frac{\partial^2 w}{\partial x\partial y},\frac{\partial^2 w}{\partial t^2},\frac{\partial^2 F}{\partial x^2},\frac{\partial^2 F}{\partial y^2},k_x,k_y,\gamma_{0k},\gamma_{1k},\gamma_{2k},\ldots,\right)$$
$$=\frac{1}{12(1-\mu^2)}\left[1-\sum_{k=1}^{N}(1-\gamma_{1k})\gamma_{0k}\right]\left[\left(\lambda^{-2}\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$
$$\left.+\left(\lambda^2\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\,\partial y}\right]$$
$$-\left[k_x\frac{\partial^2 F}{\partial y^2}+k_y\frac{\partial^2 F}{\partial x^2}\right](\cdot)-\left[1-\sum_{k=1}^{N}(1-\gamma_{2k})\gamma_{0k}\right]\frac{\partial^2 w}{\partial t^2}(\cdot)=0,$$
$$\Phi_2\left(\frac{\partial^2 F}{\partial x^2},\frac{\partial^2 F}{\partial y^2},\frac{\partial^2 F}{\partial x\partial y},\frac{\partial^2 w}{\partial t^2},\frac{\partial^2 w}{\partial x\partial y},k_x,k_y,\gamma_{0k},\gamma_{1k},\ldots,\right)$$
$$=\left[1+\sum_{k=1}^{N}\left(\frac{1}{\gamma_{1k}}-1\right)\gamma_{0k}\right]\left[\left(\lambda^2\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}\right.$$
$$\left.+\left(\lambda^{-2}\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}+2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]$$
$$+\left[k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}\right](\cdot)=0, \tag{19}$$

where N denotes the number of stiffness parts; γ_{1k}, γ_{2k} are coefficients of bending stiffness and density of the k stiffness parts [see equation (18)], showing a relative change of bending stiffness modulus and density in relation to a homogeneous shell. For a homogeneous shell $\gamma_{1k} = \gamma_{2k} = 1$. For $\gamma_{1k} < 1$ the shell is less stiff, whereas for $\gamma_{1k} > 1$ the shell is stiff. The same holds true for parameter γ_{2k}.

3. METHOD OF SOLUTION

The initial equations (19) are solved using the variational Bubnov–Galerkin method with higher approximations. The functions being sought w, F have the forms

$$w = \sum_{i,j} A_{ij}\,\varphi_{ij}(x, y),\ F = \sum_{i,j} B_{ij}\psi_{ij}(x, y), \quad i, j = 1,2,\ldots,n \tag{20}$$

Taking into account equation (20) and applying the Bubnov–Galerkin method to equation (19),

$$\int_0^1\int_0^1 \Phi_1\left[\frac{\partial^2}{\partial x^2}\sum_{i,j} A_{ij}\,\varphi_{ij}(x, y), \ldots\right]\varphi_{vz}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

$$\int_0^1\int_0^1 \Phi_2\left[\frac{\partial^2}{\partial x^2}\sum_{i,j} B_{ij}\,\Psi_{ij}(x, y), \ldots\right]\Psi_{vz}(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0, \quad i,j = 1, 2, \ldots, n. \tag{21}$$

Applying the same procedure to equation (19) in relation to the spatial co-ordinates, the following system of ordinary differential equations in relation to time and the following system of algebraic equations is obtained:

$$\sum_{v,z=1}^{n}\left[\sum_{i,j=1}^{n} \ddot{A}_{ij}(t)J^{vz}_{2,vzij} = \sum_{i,j=1}^{n} A_{ij}(t)J^{vz}_{1,vzij} + B_{vz}(t)J_{3,vzij}\right],$$

$$\sum_{v,z=1}^{n}\left[\sum_{i,j=1}^{n} B_{ij}(t)J^{vz}_{4,vzij} = A_{vz}(t)J_{5,vzij}\right], \tag{22}$$

where $\sum_{v,z=1}^{n}$ defines the a number of equations in the solving equations system, and the integrals of the Bubnov–Galerkin procedure are defined below:

$$J_{1,vzij} = \int_0^1\int_0^1 \frac{1}{12(1-\mu^2)}\left[1 - \sum_{k=1}^{N}(1-\gamma_{1k})\gamma_{0k}\right]\left[\frac{1}{\lambda^2}\frac{\partial^2\varphi_{ij}}{\partial x^2}\frac{\partial^2\varphi_{vz}}{\partial x^2}\right.$$

$$\left. + \lambda^2\frac{\partial^2\varphi_{ij}}{\partial y^2}\frac{\partial^2\varphi_{vz}}{\partial y^2} + 2(1-\mu)\frac{\partial^2\varphi_{ij}}{\partial x\partial y}\frac{\partial^2\varphi_{vz}}{\partial x\partial y} + \mu\left(\frac{\partial^2\varphi_{ij}}{\partial x^2}\frac{\partial^2\varphi_{vz}}{\partial y^2} + \frac{\partial^2 w_{ij}}{\partial y^2}\frac{\partial^2\varphi_{vz}}{\partial x^2}\right)\right]\mathrm{d}x\,\mathrm{d}y,$$

$$J_{2,vzij} = \int_0^1\int_0^1 -\left[1 - \sum_{k=1}^{N}(1-\gamma_{2k})\gamma_{0k}\right]\varphi_{ij}\varphi_{vz}\,\mathrm{d}x\,\mathrm{d}y,$$

$$J_{3,vzij} = \int_0^1 \int_0^1 -\left[k_x \frac{\partial^2 \psi_{ij}}{\partial y^2} + k_y \frac{\partial^2 \psi_{ij}}{\partial x^2}\right]\varphi_{vz}\, \mathrm{d}x\, \mathrm{d}y,$$

$$J_{4,vzij} = \int_0^1 \int_0^1 \left[1 + \sum_{k=1}^{N} \left(\frac{1}{\gamma_{1k}} - 1\right)\gamma_{0k}\right]\left[\lambda^2 \frac{\partial^2 \psi_{ij}}{\partial y^2} - \mu \frac{\partial^2 \psi_{ij}}{\partial x^2}\right)\frac{\partial^2 \psi_{vz}}{\partial y^2}$$

$$+ \left(\lambda^{-2} \frac{\partial^2 \psi_{ij}}{\partial x^2} - \mu \frac{\partial^2 \psi_{ij}}{\partial y^2}\right)\frac{\partial^2 \psi_{vz}}{\partial x^2} + 2(1+\mu)\frac{\partial^2 \psi_{ij}}{\partial x \partial y}\bigg)\frac{\partial^2 \psi_{vz}}{\partial x \partial y}\bigg]\, \mathrm{d}x\, \mathrm{d}y$$

$$J_{5,vzij} = \int_0^1 \int_0^1 -\left[k_x \frac{\partial^2 \varphi_{ij}}{\partial y^2} + k_y \frac{\partial^2 \varphi_{ij}}{\partial x^2}\right]\psi_{vz}\, \mathrm{d}x\, \mathrm{d}y. \tag{23}$$

The integrals (23) are calculated on a whole average shell surface.

A problem related to the determination of free vibration frequencies of a shell is reduced to finding the eigenvalues of the corresponding matrix. When the different stiffness parts of a shell occur, a second power of a harmonics frequency cannot be defined explicitly from the equations of shell vibration, because the equations cannot be split into the isolated equations of the corresponding harmonics. The double indices *vz* and *ij* in the sums are changed into a single indices. First, a right index is changed keeping notations for the corresponding new indices as *vz* and *ij*.

We use the following matrices:

$$A = \| A_{ij} \|,\ \ddot{A} = \| \ddot{A}_{ij} \|,\ B = \| B_{ij} \|,\ J_1 = \| J_{1,vzij} \|,$$

$$J_2 = \| J_{2,vzij} \|,\ J_3 = \| J_{3,vzij} \|,\ J_4 = \| J_{4,vzij} \|,\ J_5 = \| J_{5,vzij} \|, \tag{24}$$

where J_1–J_5 are the square matrices of order $2n$, and A, $\ddot{A}$, B denote the matrix (column) of the $2n \times 1$ type.

The solution to equations (22), using equation (24), has the following form:

$$J_2 \ddot{A} = J_1 A + J_3 B,$$

$$J_4 B = J_5 A. \tag{25}$$

From equation (25) we obtain

$$B = J_4^{-1} J_5 A. \tag{26}$$

Introducing equation (26) into the first equation (25) and multiplying the equation by J_4^{-1} we get

$$\ddot{A} = J_2^{-1} (J_1 A + J_3 J_4^{-1} J_5 A) = J_2^{-1} (J_1 + J_3 J_4^{-1} J_5) A = DA, \tag{27}$$

where

$$D = J_2^{-1} (J_1 + J_3 J_4^{-1} J_5). \tag{28}$$

The D matrix is unsymmetric when the different stiffness parts occur for which stiffness parts occur for which $\gamma_{2k} \neq 1$ (it means that a non-homogeneity of density occurs). The matrix becomes symmetric for an arbitrary values of stiffness coefficients γ_{1k}. The matrix eigenvalues are related to second power of free vibration frequencies. In order to find the eigenvalues and eigenvectors of the symmetric matrix D, we have used procedures which define all eigenvalues are eigenvectors of the symmetric three diagonal matrix. The latter is obtained from the initial symmetric matrix D using the Housholder's transformation.

In order to get all eigenvalues and eigenvectors of the real unsymmetric matrix D it can be reduced to the Hessenberg's form.

In the case of a homogeneous shell the algorithm described fully overlaps with an analytical solution (3), because in this case the system of solving equations can be split into the equations related to the different (isolated) harmonics.

4. NUMERICAL RESULTS

As an example of the theory outlined a free support (balls) of a shell on the flexible unstretched (uncompressed) rib is considered. The following boundary conditions are introduced:

$$w = M_{11} = T_{11} = \varepsilon_{11} = 0 \quad \text{for} \quad x = 0;\, 1,$$

$$w = M_{22} = T_{22} = \varepsilon_{22} = 0 \quad \text{for} \quad y = 0;\, 1, \tag{29}$$

where

$$M_{11} = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right),\; M_{22} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right),\; M_{12} = -D(1-\mu)\,\frac{\partial^2 w}{\partial x \partial y}.$$

The boundary conditions (29) can be rewritten as

$$w = w''_{xx} = F = F''_{xx} = 0 \quad \text{for} \quad x = 0;\, 1,$$

$$w = w''_{yy} = F = F''_{yy} = 0 \quad \text{for} \quad x = 0;\, 1. \tag{30}$$

In order to solve the equations (19) using the Bubnov–Galerkin method with higher approximations the expressions for the deflection $w(x, y)$ and forces $F(x, y)$ are presented in the forms

$$w = \sum_{i,j=1}^{n} A_{ij}(t) \sin i\pi x \sin i\pi y,$$

$$F = \sum_{i,j=1}^{n} B_{ij}(t) \sin i\pi x \sin i\pi y, \quad i, j = 1,2,3,\ldots,n. \tag{31}$$

The details are given in Appendix A.

The results for $n = 3$ are obtained from equation (31). It means that the dependence of the first nine modes of homogeneous and non-homogeneous squared ($\lambda = 1$) shells versus the different control parameters is investigated. All of the problems are classified in relation to a form, a number and a relative position of the stiffness parts of a shell [Figures 1(a–d)].

Case (a) corresponds to a shell with one rectangular stiffness part located in the centre. Case (b) corresponds to the stiffness part located in the shells first quarter. Case (c) is related to a part with a wide rib located symmetrically in relation to the shell's axes. Case (d) correspond to the "perforation" type: square stiffness parts are regularly located on a shell. As the control parameter, a number of those parts along one shell's side is taken.

4.1. A HOMOGENEOUS SHELL

In Figures 2(a, b), a relation between the frequency of a spherical shell with a rectangular plane versus curvature of a non-dimensional shell $k_x = k_y = 0 - 36$ is given. The curves 1, 2, 4, 5, 7, 9 correspond to the mode numbers. Increasing the curvature shell's parameter all of the modes increase monotonously. The lower modes are more sensitive to a change of this parameter.

4.2. A NON-HOMOGENEOUS SHELL

During investigation of a non-homogeneous shell vibration for each frequency the coefficient K_d, describing the harmonics dynamics, is introduced. It is defined as

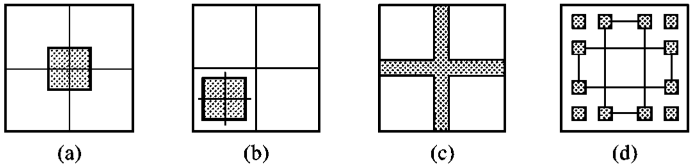


Figure 1. Location of non-homogeneous parts of stiffness on a shell: (a) square part located in the centre; (b) square part located in the quarter; (c) flexural stiffness changes along the axes; (d) "perforation" type.

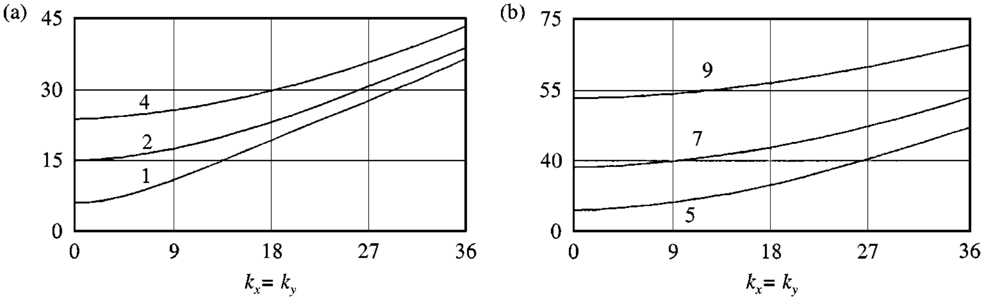


Figure 2. Frequencies versus the curvatures of the shell.

a relation between the corresponding frequency of non-homogeneous to a homogeneous shell.

In Figures 3 and 4, the relationship between K_d and a surface of a central stiffness part for a plate ($k_x = k_y = 0$) and a shell ($k_x = k_y = 36$) are given. The curve number corresponds to a mode number. The curves lying below that with number 1 correspond to a more soft plate and shell ($\gamma_{1k} = 0{,}5$; $\gamma_{2k} = 1$). The curves lying above that curve correspond to a more stiff one ($\gamma_{1k} = 1{,}5$; $\gamma_{2k} = 1$) in comparison to a homogeneous plate and shell (for a homogeneous plate and shell we have $\gamma_{1k} = \gamma_{2k} = 1$).

As it can be seen from the figures, the coefficient K_d of all the modes of a plate or a shell in a stiff (soft) case increases (decreases) monotonously with an increase in the heterogeneous surface S. In a limiting case, when a whole shell becomes non-homogeneous, an increase of K_d reaches 25–30%.

In Figure 5 and 6, similar dependencies are given as in the (b) of Figure 1. The character of the curves is an analogical one, but an increase of the K_d coefficient does not achieve 10%. This means that an occurrence of non-homogeneity of a shell located in its quarter exhibits a lower influence on the frequencies of the shell, than with the non-homogeneity located in the centre.

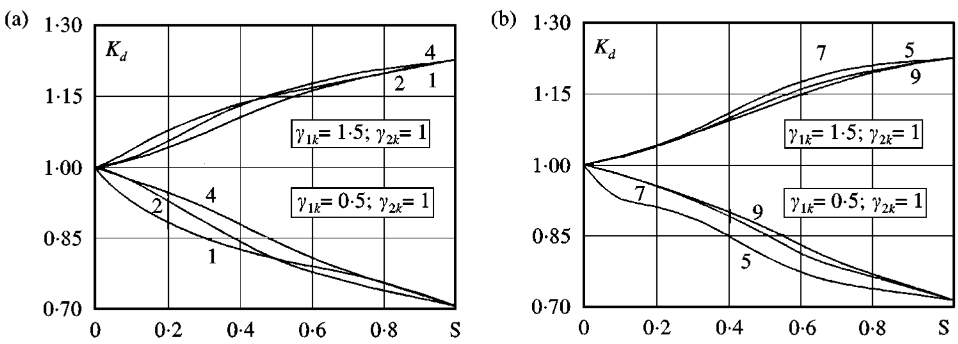


Figure 3. Dynamical modes plate coefficient ($k_x = k_y = 0$) versus a central stiffness parts surface [see Figure 1(a)].

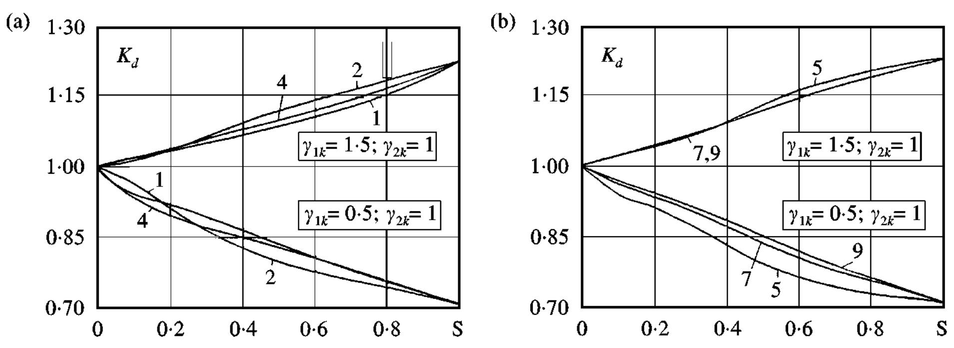


Figure 4. Dynamical modes shell coefficient ($k_x = k_y = 36$) versus a surface of the square stiffness part located in the centre [see Figure 1(a)].

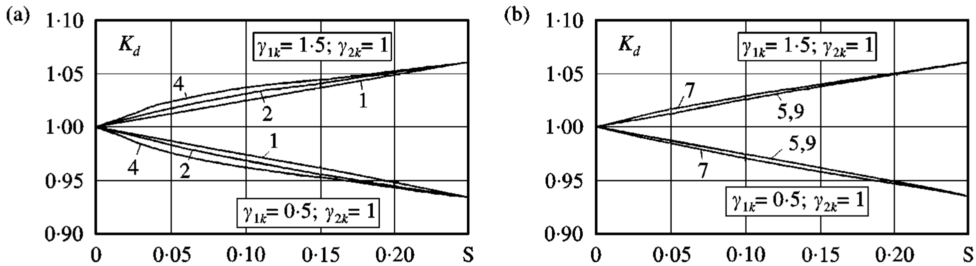


Figure 5. Dynamical modes plate coefficient ($k_x = k_y = 0$) versus a surface of the stiffness part located in the quadrant [see Figure 1(b)].

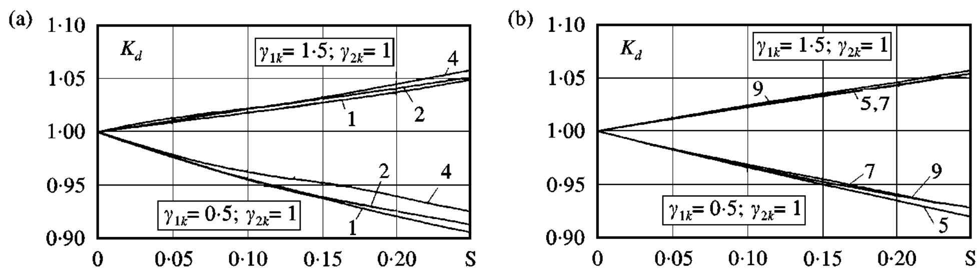


Figure 6. Dynamical modes shell coefficient ($k_x = k_y = 36$) versus a surface of the stiffness part located in the quarter [see Figure 1(b)].

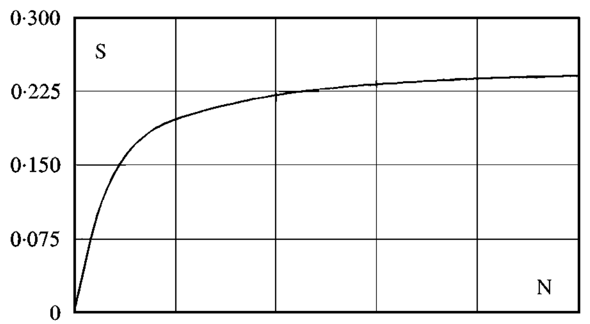


Figure 7. General surface stiffness versus the parts number along one shell's side [see (Figure 1(d) the "perforation" case].

For a perforation type non-homogeneity [case (d) in Figure 1], note that the perforation covers the shell in a regular way, and a square side is equal to the distance between the successive parts. The stiffness parameters are the same for all parts. In this case it is appropriate to introduce the N parameter which corresponds to the parts number along one of the shell's side and then to investigate a general surface non-homogeneity. This relationship is shown in Figure 7.

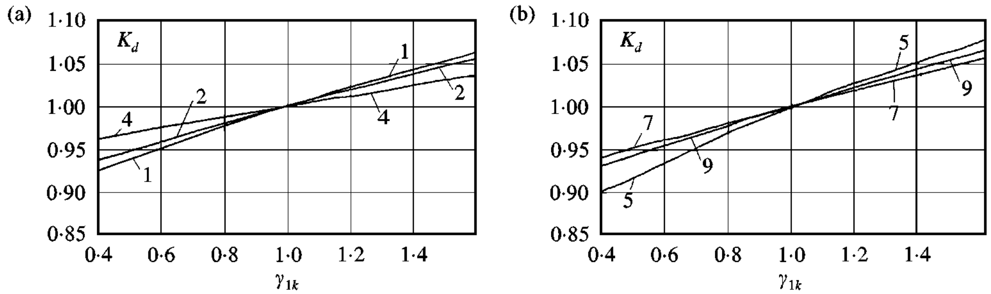


Figure 8. Dynamical modes shell coefficient ($k_x = k_y = 0$) versus stiffness part coefficient [see Figure 1(c); the ribb's wide 0,1)].

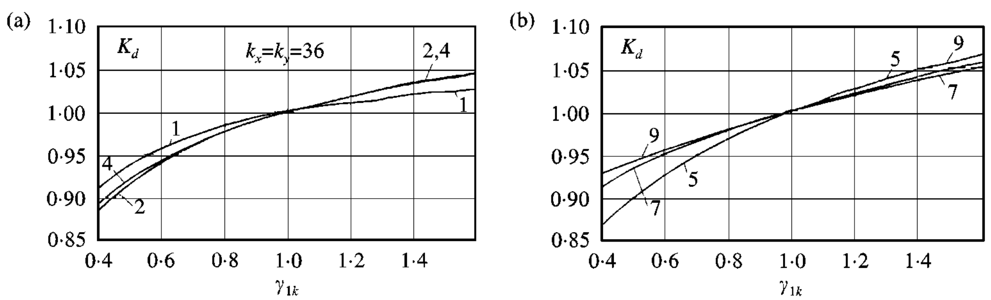


Figure 9. Dynamical modes shell coefficient ($k_x = k_y = 36$) versus stiffness part coefficient [see Figure 1(c); the ribb's wide 0,1)].

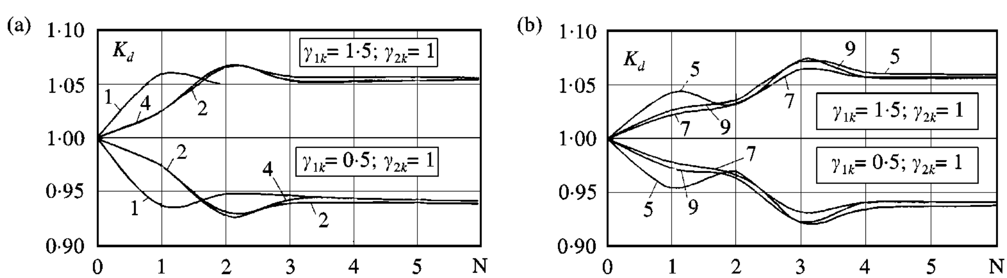


Figure 10. Dynamical modes plate coefficient ($k_x = k_y = 0$) versus the stiffness parts number N along one of the plates side [see Figure 1(d)].

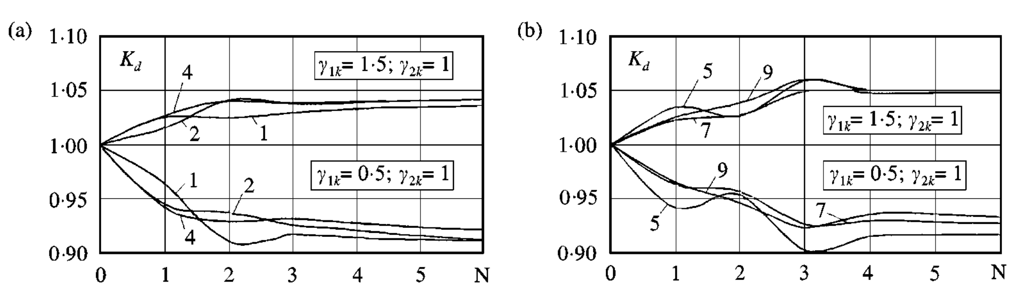


Figure 11. Dynamical modes shell coefficient ($k_x = k_y = 36$) versus the stiffness parts number N along one of the plate's side [see Figure 1(d)].

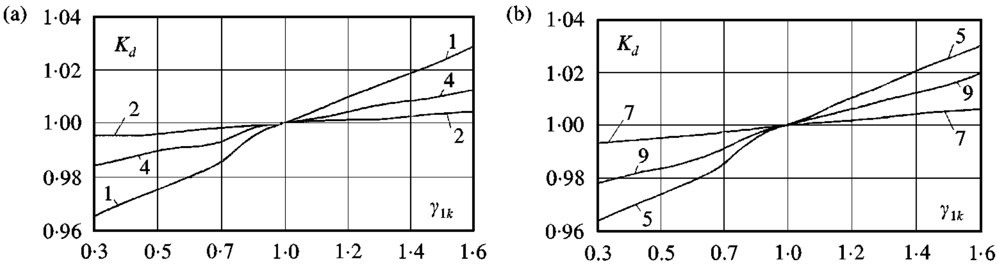
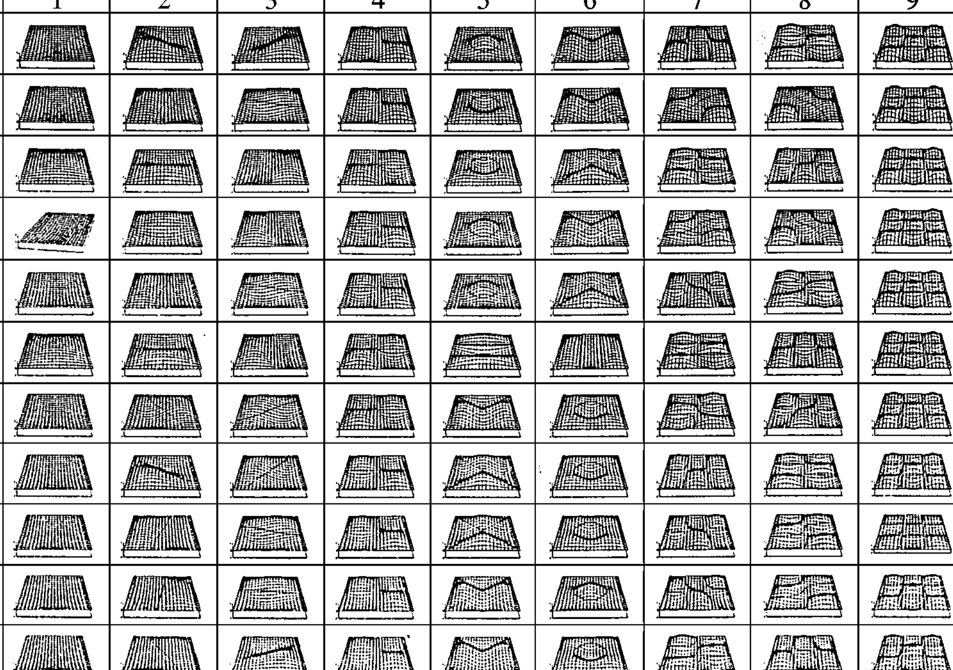


Figure 12. Dynamical modes plate coefficient ($k_x = k_y = 0$) versus the stiffness coefficient of the central stiffness part [see Figure 1(a)].

Table 1

Vibration modes of a non-homogeneous [see Figure 1(a)] shell ($k_x = k_y = 36$) versus the stiffness coefficient γ_{1k}, $\gamma_{2k} = 1$ of the part with the co-ordinates: $x_1 = y_1 = 0,4$; $y_1 = 0,6$



	Mode								
γ_{1j}	1	2	3	4	5	6	7	8	9
0,5									
0,6									
0,7									
0,8									
0,9									
1,0									
1,1									
1,2									
1,3									
1,4									
1,5									

In Figure 8 and 9 the K_d relationships for a plate ($k_x = k_y = 0$) and for a shell ($k_x = k_y = 36$) and a stiffness coefficient γ_{1k} for a fixed wide rib (0, 1) and $\gamma_{2k} = 1$ [case (c) of Figure 1] is given. The same earlier notations are used. It should be noted that in the range of γ_{1k} considered, the changes in the dependence for a plate is practically linear.

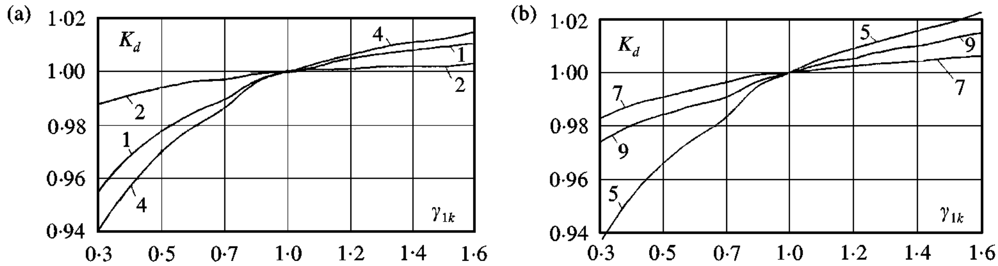
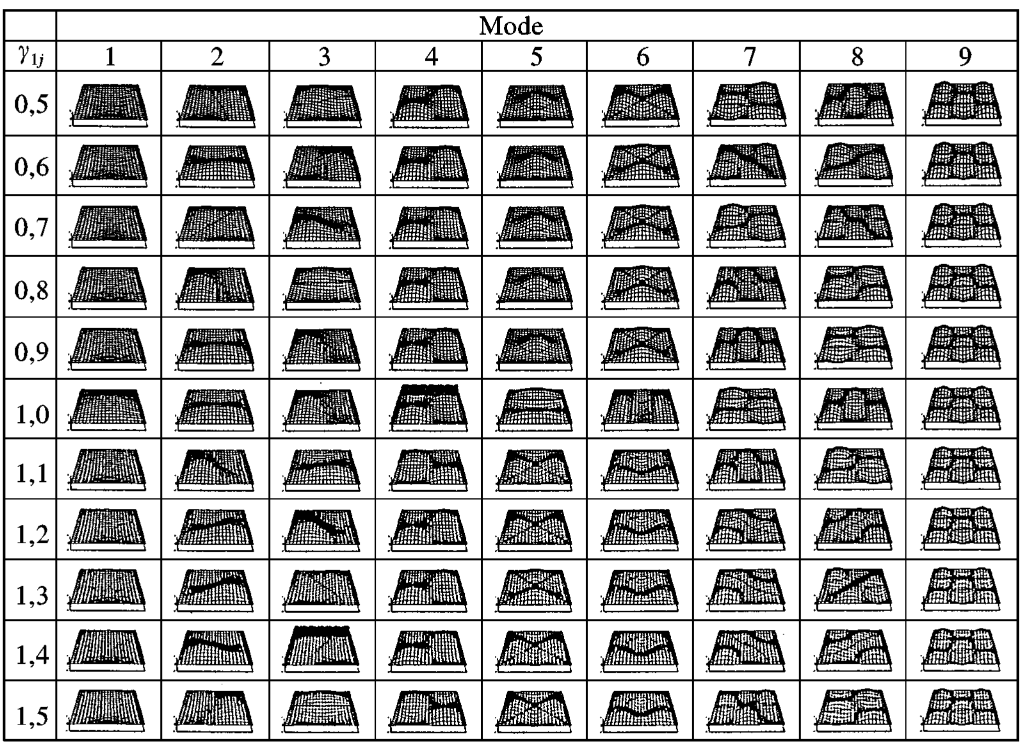


Figure 13. Dynamical modes shell coefficient ($k_x = k_y = 36$) versus the stiffness coefficient of the central stiffness part [see Figure 1(a)].

TABLE 2

Vibration modes of a non-homogeneous [see Figure 1(a)] plate ($k_x = k_y = 0$) versus the stiffness coefficient γ_{1k}, $\gamma_{2k} = 1$ of the part with the co-ordinates: $x_1 = y_1 = 0{,}4$; $y_1 = y_2 = 0{,}6$

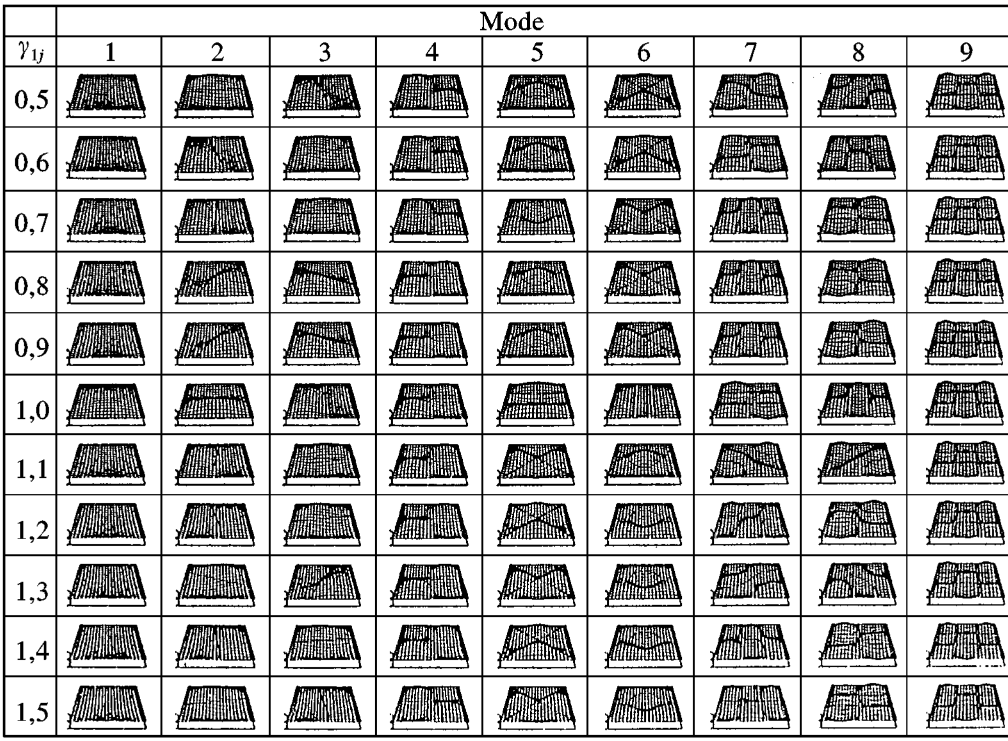


	Mode								
γ_{1j}	1	2	3	4	5	6	7	8	9
0,5									
0,6									
0,7									
0,8									
0,9									
1,0									
1,1									
1,2									
1,3									
1,4									
1,5									

For a shell, the relationship has a non-linear character which influences more higher modes. It is seen that an increase of the N parameter above 8 does not change the dependence substantially. Therefore, a stabilization of relations are expected for those cases.

TABLE 3

Vibration modes of a non-homogeneous [*see Figure* 1(a)] *shell* ($k_x = k_y = 18$) *versus the stiffness coefficient* γ_{1k}, $\gamma_{2k} = 1$ *of the part with the co-ordinates*: $x_1 = y_1 = 0{,}4$; $y_1 = y_2 = 0{,}6$



In Figures 10 and 11 the relationship between the coefficient K_d and N for a plate ($k_x = k_y = 0$) and for a shell ($k_x = k_y = 36$) are reported. The same notation considered in previous cases is used. The curves below 1 correspond to softer plate and shell with $\gamma_{1k} = 0{,}5$; $\gamma_{2k} = 1$. The curves lying higher than that denoted by 1 correspond to those stiffer ($\gamma_{1k} = 1{,}5$; $\gamma_{2k} = 1$) than homogeneous plate and shell.

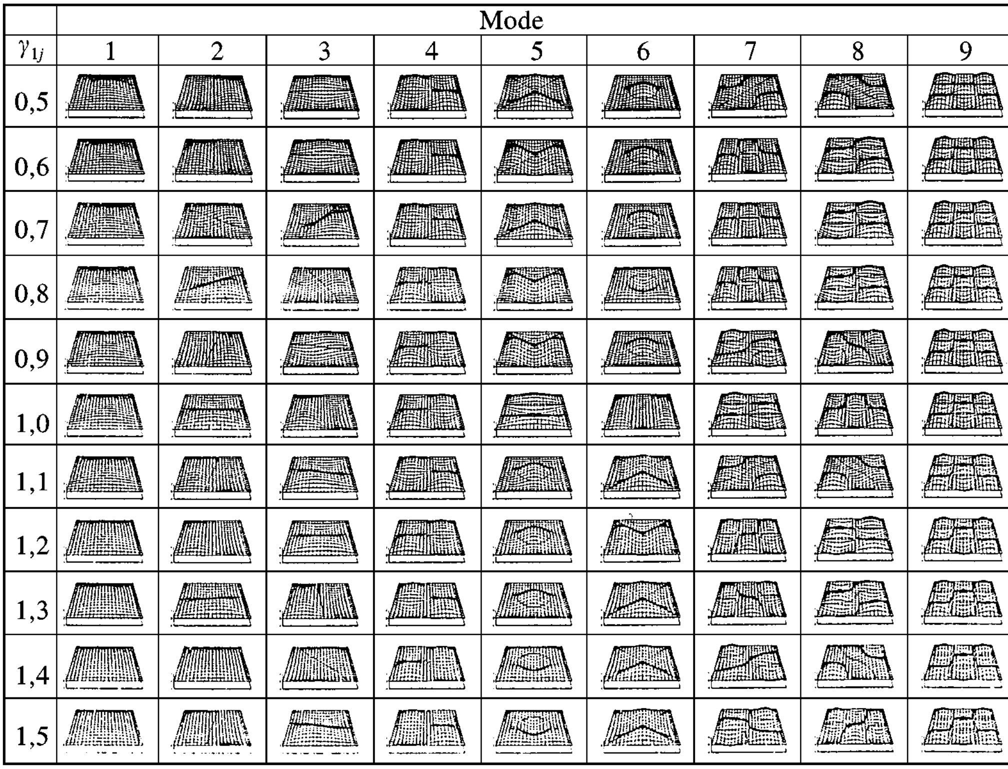
Contrary to the relationships considered earlier the curves are more complicated. A zone exists with distinct local maxima as well as a zone of coefficient K_d stabilisation. In the first case, for certain values of the parameter ($N = 1, 2, 3$) the sum of the surface of non-homogeneity has a critical value. In the second case ($N > 4$), an increase in the non-homogeneous surface does not in fact influence the relationship because the increase of the surface is small.

Figures 12 and 13 the relationship between the dynamic modes coefficient of the non-homogeneous plate [Figure 1(a)] and the non-homogeneous shell ($k_x = k_y = 36$) and the stiffness γ_{1k} ($\gamma_{2k} = 1$) are shown.

Analysis of the figures indicates that a stiffness coefficient influences the dynamic modes coefficient of a shell more than of a plate (it does not exceeds 5%).

TABLE 4

Vibration modes of a non-homogeneous [see Figure 1(c)] shell ($k_x = k_y = 36$) versus the stiffness coefficient γ_{1k}, $\gamma_{2k} = 1$ of the part with the wide rib 0,1

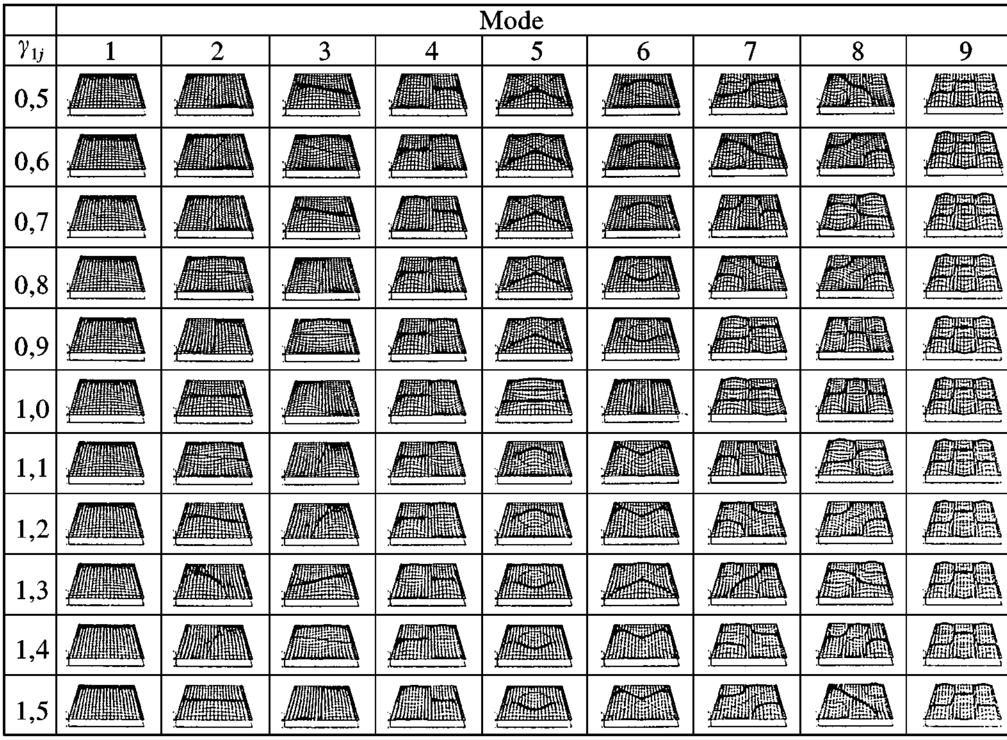


In Tables 1–6 the form of homogeneous ($\gamma_{1k} = 1, \gamma_{2k} = 1$) and non-homogeneous ($\gamma_{1k} \neq 1, \gamma_{2k} = 1$) plate ($k_x = k_y = 0$) and shells ($k_x = k_y = 18$; 36) for all nine modes versus γ_{1k} are presented. The non-homogeneous form of Figure 1(a) corresponds to Tables 1–3 with the non-homogeneous part co-ordinates $x_1 = y_1 = 0{,}4$; $x_2 = y_2 = 0,\ 6$. Tables 4–6 correspond to the non-homogeneous schematic of Figure 1 with the wide rib of 0.1. For comparison purposes, a case of homogeneous plate or shell ($\gamma_{1k} = 1$) is also included.

The mechanical approach used is not able to take account of any local phenomena at the interface between two adjacent zone of the considered non-homogeneous materials.

TABLE 5

Vibration modes of a non-homogeneous [see Figure 1(c)] *plate* ($k_x = k_y = 0$) *versus the stiffness coefficient* γ_{1k}, $\gamma_{2k} = 1$ *of the part with the wide rib* 0,1



ACKNOWLEDGMENT

This work has been supported by the Polish State Committee for Scientific Research (grant No 8T11F 014 16).

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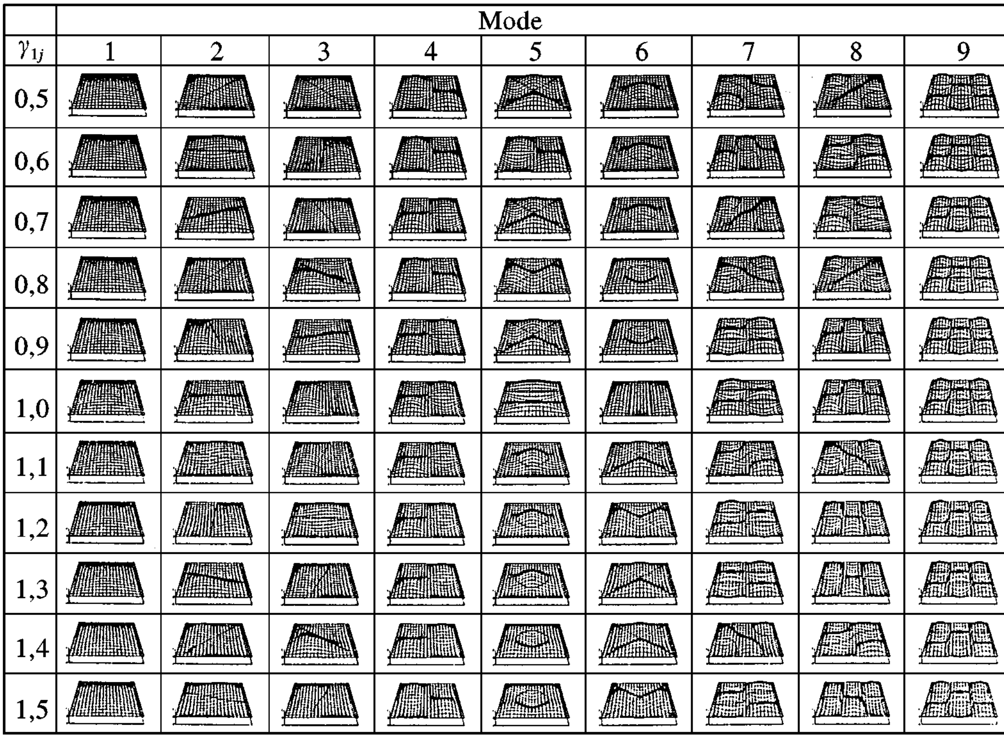
TABLE 6

Vibration modes of a non-homogeneous [see Figure 1(c)] shell ($k_x = k_y = 18$) versus the stiffness coefficient γ_{1k}, $\gamma_{2k} = 1$ of the part with the wide rib 0,1



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APPENDIX A: INTEGRALS OF THE BUBNOV–GALERKIN PROCEDURE

The Bubnov–Galerkin procedure is applied to equation (19) in relation to the spatial co-ordinates. As a result we obtain the system of differential equation in relation to time and the system of the algebraic equations

$$\sum_{v,z=1}^{n}\left[\sum_{i,j=1}^{n}\ddot{A}_{ij}(t)\,J_{2,vzij}^{vz}=\sum_{i,j=1}^{n}A_{ij}(t)\,J_{1,vzij}^{vz}+B_{vz}(t)\,J_{3,vzij}^{vz}\right],$$

$$\sum_{v,z=1}^{n}\left[\sum_{i,j=1}^{n}B_{ij}(t)\,J_{4,vzij}^{vz}=A_{vz}(t)\,J_{3,vzij}^{vz}\right], \tag{A.1}$$

where $\sum_{v,z=1}^{n}$ defines the number of equations. Below are given the integrals of the Bubnov–Galerkin procedure:

$$I_{1,vi}=\int_0^1\sin i\pi x\sin v\pi x\,\mathrm{d}x=\begin{cases}\frac{1}{2}, & i=v\\ 0, & i\neq v\end{cases},\quad I_{2,zj}=\int_0^1\sin j\pi x\sin z\pi x\,\mathrm{d}y=\begin{cases}\frac{1}{2}, & j=z\\ 0, & j\neq z\end{cases},$$

$$I_{3,vi}=\int_0^1\cos i\pi x\cos v\pi x\,\mathrm{d}x=\begin{cases}\frac{1}{2}, & i=v\\ 0, & i\neq v\end{cases},\quad I_{4,zj}=\int_0^1\cos j\pi x\cos z\pi x\,\mathrm{d}y=\begin{cases}\frac{1}{2}, & j=z\\ 0, & i\neq z\end{cases},$$

$$I_{5,vi}=\int_0^1\gamma_{0k}\sin i\pi x\sin v\pi x\,\mathrm{d}x$$

$$=\begin{cases}\left[\dfrac{\sin(i-v)\pi x_2-\sin(i-v)\pi x_1}{2\pi(i-v)}-\dfrac{\sin(i+v)\pi x_2-\sin(i+v)\pi x_1}{2\pi(i+v)}\right], & i\neq v\\[2ex] \left[\dfrac{x_2-x_1}{2}-\dfrac{\sin 2i\pi x_2-\sin 2i\pi x_1}{4i\pi}\right], & i=v\end{cases}$$

$$I_{6,zj} = \int_0^1 \gamma_{0k} \sin i\pi y \sin z\pi y \, \mathrm{d}y$$

$$= \begin{cases} \left[\dfrac{\sin(j-z)\pi y_2 - \sin(i-z)\pi y_1}{2\pi(j-z)} - \dfrac{\sin(j+z)\pi y_2 - \sin(j+z)\pi y_1}{2\pi(j+z)} \right], & j \neq z \\ \left[\dfrac{y_2 - y_1}{2} - \dfrac{\sin 2j\pi y_2 - \sin 2j\pi y_1}{4j\pi} \right], & j = z, \end{cases}$$

$$I_{7,vi} = \int_0^1 \gamma_{0k} \cos i\pi x \cos nvz\pi x \, \mathrm{d}x$$

$$= \begin{cases} \left[\dfrac{\sin(i-v)\pi x_2 - \sin(i-v)\pi x_1}{2\pi(i-v)} + \dfrac{\sin(i+v)\pi x_2 - \sin(i+v)\pi x_1}{2\pi(i+v)} \right], & i \neq v \\ \left[\dfrac{x_2 - x_1}{2} + \dfrac{\sin 2i\pi x_2 - \sin 2i\pi x_1}{4i\pi} \right], & i = v \end{cases}$$

$$I_{8,zj} = \int_0^1 \gamma_{0k} \cos i\pi y \cos z\pi y \, \mathrm{d}y$$

$$= \begin{cases} \left[\dfrac{\sin(j-z)\pi y_2 - \sin(j-z)\pi y_1}{2\pi(j-z)} - \dfrac{\sin(j+z)\pi y_2 - \sin(j+z)\pi y_1}{2\pi(j+z)} \right], & j \neq z \\ \left[\dfrac{y_2 - y_1}{2} - \dfrac{\sin 2j\pi y_2 - \sin 2j\pi y_1}{4j\pi} \right], & j = \mathrm{z} \end{cases}$$

$$J_{1,vzij} = -\frac{\pi^4}{12(1-\mu^2)} \sum_{k=1}^{N} (1-\gamma_{1k}) \left[\left(\frac{i^2}{\lambda^2} + \mu j^2 \right) v^2 I_{5,vi} I_{6,zj} \right.$$

$$\left. + (\lambda^2 j^2 + \mu i^2) z^2 I_{5,vi} I_{6,zj} + 2(1-\mu) ijvz I_{7,vi} I_{8,zj} \right],$$

$$J^{vz}_{1,vzij} = J_{1,vzij} + \frac{\pi^4}{12(1-\mu^2)} \left[\left(\frac{v^4}{\lambda^2} + \lambda^2 v^4 \right) I_{1,vi} I_{2,zj} + 2v^2 z^2 I_{3,vi} I_{4,zj} \right],$$

$$J_{2,vzij} = \sum_{k=1}^{N} (1-\gamma_{2k}) I_{5,vi} I_{6,zj},$$

$$J^{vz}_{2,vzij} = J_{2,vzij} - I_{1,vi} I_{2,zj},$$

$$J_{3,vzij} = (k_x z^2 + k_y v^2) \pi^2 I_{1,vi} I_{2,zj},$$

$$J_{4,vzij} = \pi^4 \sum_{k=1}^{N} \left(\frac{1}{\gamma_{1k}} - 1\right)\Bigg[(\lambda^2 j^2 + \mu i^2) z^2 I_{5,vi} I_{6,zj}$$

$$+ \left(\frac{i^2}{\lambda^2} - \mu j^2\right) v^2 I_{5,vi} I_{6,zj} + 2(1+\mu) ijvz I_{7,vi} I_{8,zj}\Bigg],$$

$$J_{4,vzij}^{vz} = J_{4,vzij} + \pi^4 \left[\left(\frac{v^4}{\lambda^2} + \lambda^2 z^4\right) I_{1,vi} I_{2,zj} + 2v^2 z^2 I_{3,vi} I_{4,zj}\right]. \tag{A.2}$$